NONLINEAR LAGRANGIANS OF THE RICCI TYPE [1]

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Abstract

The Euler-Lagrange equations for some class of gravitational actions are calculated by means of Palatini principle. Polynomial structures with Einstein metrics appear among extremals of this variational problem.

1. INTRODUCTION

A *polynomial structure* on an n-dimensional differentiable manifold M is given by type $(1,1)$ tensor field $S \equiv S^{\mu}_{\nu}$ of constant rank r $(1 \leq r \leq n)$, which satisfies polynomial equation $\pi(S) = 0$ for some polynomial $\pi(t)$ of real coefficients. Almost-complex and almost-product structures are among the best known examples and the most fundamental structures of this kind [1]. It has been recently shown that both these structures appear in a natural way from the first-order (Palatini) variational principle applied to general class of non-linear Lagrangians depending on the Ricci squared invariant constructed out of a metric and a symmetric connection [2]. Moreover, Einstein equations of motion and Komar energy-momentum complex are *universal* for this class of Lagrangians [3]. The non-linear gravitational Lagrangians which still generate Einstein equations are particularly important since, at the classical level, they are equivalent to General Relativity. However, their quantum contents and divergences could be slightly improved.

In the present note, we are going to extend above results showing that more general Ricci type Lagrangians lead to more general polynomial structures and that the universality property remains still valid; both for the equations as for the energy-momentum. The techniques used here for analysis of the Euler-Lagrange equations are similar to the ones applied in [3, 4, 5] (c.f. [6] for summary). A different approach that missed polynomial relations has been recently proposed in [7].

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1.1. Preliminaries and Notation

Einstein metrics are extremals of the Einstein-Hilbert purely metric variational problem. It is known that the non-linear Einstein-Hilbert type Lagrangians $f(R)\sqrt{g}$, where f is a function of one real variable and R is a scalar curvature of a metric g [2], lead to fourth order equations for g which are not equivalent to Einstein equations unless $f(R) = R - c$ (linear case), or to appearance of additional matter fields. It is also known that the linear "first order" Lagrangian $r\sqrt{g}$, where $r = r(g,\Gamma) = g^{\alpha\beta}r_{\alpha\beta}(\Gamma)$ is a scalar concomitant of the metric g and linear (symmetric) connection Γ, [3] leads to separate equations for g and Γ which turn out to be equivalent to the Einstein equations for g (so-called Palatini principle, c.f. [8, 9, 10, 11, 12]).

In the sequel we shall use lower case letters $r^{\alpha}_{\beta\mu\nu}$ and $r_{\beta\nu} = r^{\alpha}_{\beta\alpha\nu}$ to denote the Riemann and Ricci tensor of an arbitrary (symmetric) connection Γ

$$\begin{aligned} r^{\alpha}_{\beta\mu\nu} &= r^{\alpha}_{\beta\mu\nu}(\Gamma) = \partial_\mu \Gamma^{\alpha}_{\beta\nu} - \partial_\nu \Gamma^{\alpha}_{\beta\mu} + \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu} \\ r_{\mu\nu} &= r_{\mu\nu}(\Gamma) = r^{\alpha}_{\mu\alpha\nu} \end{aligned} \tag{1.1}$$

i.e. without assuming that Γ is the Levi-Civita connection of g.

Unlike in a purely metric case, an equivalence with General Relativity also holds for non-linear gravitational Lagrangians

$$L_f(g,\Gamma) = \sqrt{g}\, f(r) \tag{1.2}$$

(parameterized by the real function f of one variable), when they were considered within the first-order Palatini formalism [4]. Similar analysis were performed for "Ricci squared" non-linear Lagrangians

$$\hat{L}_f(g,\Gamma) = \sqrt{g}\, f(s) \tag{1.3}$$

where, $s = s(g,\Gamma) = g^{\alpha\mu}g^{\beta\nu}r_{(\alpha\beta)}r_{(\mu\nu)}$, and $r_{(\mu\nu)} = r_{(\mu\nu)}(\Gamma)$ is the symmetric part of the Ricci tensor of Γ. (Thereafter () denotes a symmetryzation.)

Let us consider a $(1,1)$ tensor valued concomitant of a metric g and a linear torsionless connection Γ defined by

$$S^{\mu}_{\nu} \equiv S^{\mu}_{\nu}(g,\Gamma) = g^{\mu\lambda}r_{(\lambda\nu)}(\Gamma) \tag{1.4}$$

One can use it to define a family of scalar concomitants of the Ricci type

$$s_k = \mathtt{tr} S^k \tag{1.5}$$

for $k = 1, \ldots n$. We can eliminate the higher order Ricci scalars s_k with $k > n$, by using a characteristic polynomial equation for the $n \times n$ matrix S (c.f. [7]). One immediately recognizes that $r \equiv s_1 = \mathtt{tr} S$ and $s \equiv s_2 = \mathtt{tr} S^2$.

[2] One simply writes $\sqrt{g}$ for $\sqrt{|\mathtt{det} g|}$.

[3] Now, the scalar $r(g,\Gamma) = g^{\alpha\beta}r_{\alpha\beta}(\Gamma)$ is not longer the scalar curvature, since Γ is not longer the Levi-Civita connection of g.

2. NONLINEAR RICCI LAGRANGIANS

Our goal in the present note is to apply a Palatini principle to the more general family of non-linear gravitational Lagrangians of the Ricci type

$$L_F(g,\Gamma) = \sqrt{g}\, F(s_1, \ldots, s_n) \tag{2.1}$$

parameterized by the real-valued function F of n-variables. This family includes the previous ones as particular cases.

2.1. Equation of Motion

According to the Palatini prescription, we choose a metric g and a symmetric connection Γ on a space-time manifold M as independent dynamical variables. Variation of L_F gives

$$\delta L_F = \sqrt{g}\,((\delta_g F)_{\alpha\beta} - \frac{1}{2} F g_{\alpha\beta})\, \delta g^{\alpha\beta} + \sqrt{g}\, \delta_\Gamma F \tag{2.2}$$

where obviously $\delta F = \sum_{k=1}^{n} F'_k\, \delta s_k$, and $F'_k = \frac{\partial F}{\partial s_k}$. We see at once that

$$\delta_g s_k = k\, \mathtt{tr}(S^{k-1} \delta_g S) = k\, (S^{k-1})^{\sigma}_{\alpha}\, r_{(\beta\sigma)}\, \delta g^{\alpha\beta}$$

which is clear from $\delta s_k = k\, \mathtt{tr}(S^{k-1} \delta S)$. Accordingly

$$\delta_g F = ||F'(S)||^{\sigma}_{\alpha}\, r_{(\beta\sigma)} \delta g^{\alpha\beta}$$

where for abbreviation we have introduced a $(1,1)$ tensor field concomitant

$$||F'(S)|| = \sum_{k=1}^{n} k\, F'_k\, S^{k-1} \tag{2.3}$$

In a similar manner one calculates

$$\delta_\Gamma F = ||F'(S)||^{\alpha}_{\sigma}\, g^{\sigma\beta}\, \delta r_{(\alpha\beta)} \equiv ||F'(S)||^{\alpha\beta}\, \delta r_{(\alpha\beta)} \tag{2.4}$$

where the inverse metric g^{-1} has been used for rising the lower index in $||F'(S)||$. Substituting all necessary terms into formula (2.2) gives

$$\delta L_F = \sqrt{g}\,(||F'(S)||^{\sigma}_{\alpha}\, r_{(\beta\sigma)} - \frac{1}{2} F g_{\alpha\beta})\, \delta g^{\alpha\beta} - \sqrt{g}\, ||F'(S)||^{\alpha\beta}\, \delta r_{(\alpha\beta)} \tag{2.5}$$

Taking into account the well-known Palatini formula

$$\delta r_{(\alpha\beta)} = \nabla_\mu \delta\Gamma^{\mu}_{\alpha\beta} - \nabla_{(\alpha} \delta\Gamma^{\sigma}_{\beta)\sigma}$$

with ∇_α being the covariant derivative with respect to Γ and performing the "covariant" Leibniz rule one gets the variational decomposition formula

$$\begin{aligned} \delta L_F \;\; = \;\; & \sqrt{g}\,(||F'(S)||^{\sigma}_{\alpha}\, r_{(\beta\sigma)} - \frac{1}{2} F g_{\alpha\beta})\, \delta g^{\alpha\beta} - \nabla_\nu\, [\sqrt{g}\,(||F'(S)||^{\alpha\beta}\, \delta^{\nu}_{\lambda} \\ & - ||F'(S)||^{\nu\alpha}\, \delta^{\beta}_{\lambda})]\, \delta\Gamma^{\lambda}_{\alpha\beta} + \partial_\mu\, [\sqrt{g}\, ||F'(S)||^{\alpha\beta}\, (\delta\Gamma^{\mu}_{\alpha\beta} - \delta^{\mu}_{(\beta}\, \delta\Gamma^{\sigma}_{\alpha)\sigma})] \end{aligned} \tag{2.6}$$

This formula splits δL_F into the Euler-Lagrange part and the boundary term which shall be used later on for a conserved current construction.

Therefore, the Euler-Lagrange field equations read as follows

$$||F'(S)||^{\sigma}_{(\alpha}\, r_{(\beta)\sigma)} - \frac{1}{2} F\, g_{\alpha\beta} = 0 \tag{2.7}$$

$$\nabla_\nu\, [\sqrt{g}\, (||F'(S)||^{(\alpha\beta)}\, \delta^\nu_\lambda - ||F'(S)||^{\nu(\alpha}\, \delta^{\beta)}_\lambda)] = 0 \tag{2.8}$$

Before proceeding further, it is convenient to introduce a $(0, 2)$ symmetric tensor field

$$h_{\alpha\beta} = r_{(\alpha\beta)}(\Gamma) \tag{2.9}$$

which will be extremely useful for studying symmetry properties of $||F'(S)||$. For this purpose we shall employ a matrix notation. For example: $S = g^{-1}\, h$ with both g and h being symmetric matrices (c.f. equation (1.4)), easily implies that $h\, S^k = g\, S^{k+1}$ and $S^k\, g^{-1} = S^{k+1}\, h^{-1}$ (provided that h^{-1} exists) are also symmetric matrices for arbitrary $k = 0, 1, \ldots$. Indeed since e.g. $h\, S^k = h\, g^{-1} \ldots g^{-1} h$ then it is self-transpose. In particular, $h\, ||F'(S)||$ in (2.7) and $||F'(S)||\, g^{-1}$ in (2.8) (c.f. (2.4) and (2.11)) are symmetric. In other words e.g., the matrix concomitant

$$||F'(S)||^{\alpha\beta} \equiv ||F'(S)||^\alpha_\sigma\, g^{\sigma\beta}$$

is symmetric. These properties allow us to transform the Euler-Lagrange equations (2.7-2.8) into the form

$$S\, ||F'(S)|| = \frac{1}{2} F\, I \tag{2.10}$$

$$\nabla_\nu\, (\sqrt{g}\, ||F'(S)||^{\alpha\beta}) = 0 \tag{2.11}$$

where I is a $n \times n$ identity matrix. (Compare for similar calculations presented e.g. in [3-6,13,14].)

Equations (2.10) must be considered together with a consistency condition obtained by taking the trace of (2.10). It gives

$$\sum_{k=1}^{n} k\, F'_k\, s_k = \frac{n}{2}\, F \tag{2.12}$$

The last equation (except the case it is identically satisfied) becomes a single (non-algebraic in general) equation on possible values of the Ricci scalars (remember that F and F'_k are given functions of the variables $s_1, \ldots, s_n$). It forces $(s_1, \ldots, s_n)$ to take a set of constant values $s_i = c_i$, with $(c_1, \ldots, c_n)$ being a solution of (2.12). Substituting back these constant roots into equation (2.10) we obtain a polynomial equation for the matrix S. It means that with any set $c_1, \ldots, c_n$ of the (numerical) solutions of (2.12), one can associate a polynomial

$$\pi_{c_1,\ldots,c_n}(t) = \sum_{k=1}^{n} a_k t^k \tag{2.13}$$

with constant coefficients $a_k = k\,\frac{\partial F}{\partial s_k}(c_1, \ldots, c_n)$. In other words, a lacking of an explicit dependence on a point $x \in M$ in equation (2.12), implies that the coefficients a_i are also x-independent. The above arguments can be reinforce, following the line developed in [7]: by using the characteristic equation techniques, one is allowed to introduce a complementary system of $(n-1)$-equations that additionally relate values of the Ricci scalars and which still do not depend on a point $x \in M$. Thus, instead of the single equation (2.12) we can have at our disposal a system of n-equations with n-unknowns that provides us, in a regular case, in a set of numerical (i.e. constant) solutions $(c_1, \ldots, c_n)$. But this rather technical point will be consider in more details elsewhere [15].

In this way we are led to the polynomial structure that has been defined at the very beginning. In our case the polynomial equation for S takes the form

$$S\,\pi_{c_1,\ldots,c_n}(S) = I \tag{2.14}$$

This becomes now a substitute of (2.10). (In fact, in order to get (2.14) one eventually should rescale the coefficients in (2.13) by a constant factor.) Particularly, (2.14) implies that the determinant of S is a constant. As a consequence the determinant of g is up to a constant factor proportional to that of h.

From now on unless otherwise stated we assume that S is an invertible matrix (nondegenerate case) with, of course, $S^{-1} = \pi_{c_1,\ldots,c_n}(S)$. Thus, replacing $\mathtt{det}\, g$ in (2.11) by $\mathtt{det}\, h$ and making use of the Ansätz (2.9) with $h^{-1} = \pi(S)\, g^{-1}$ (c.f. (2.4)), gives

$$\nabla_\lambda(\sqrt{h} h^{\alpha\beta}) = 0$$

with $h^{\alpha\beta}$ being the inverse of $h_{\alpha\beta}$. This, in turn, in any dimension $n > 2$ [4], forces Γ to be the Levi-Civita connection of h. Replacing back into (2.9) we find

$$h_{\mu\nu} = r_{(\mu\nu)}(\Gamma_{LC}(h)) = R_{\mu\nu}(h) \tag{2.15}$$

the Einstein equations for the metric h. Here a value of the cosmological constant is 1 due to the "unphysical" normalization made in (2.14). This shows that the use of Palatini formalism leads to results essentially different from the metric formulation when one deals with non-linear Ricci type Lagrangians: with the exception of special ("non-generic") cases we always obtain the Einstein equations as gravitational field equations. In this sense non-linear theories are equivalent to General Relativity (see also [16] in this context). They admit alternative Lagrangians for the Einstein equations with a cosmological constant.

2.2. Symmetries and Superpotentials

Though the understanding of the energy of gravitational field has not been attained yet, we can analyse the Noether symmetries and the corresponding conservation laws. Our

[4] See [5, 3] for $n = 2$ case.

Lagrangians are *reparameterization invariant*, in the sense that under a 1-parameter group of diffeomorphisms generated by an arbitrary vector field $\xi = \xi^\alpha \partial_\alpha$ on M, the Lagrangian L_F transform as a scalar density of weight 1. At the infinitesimal level, variations of the field variables are represented by the Lie derivatives $\mathcal{L}_\xi$, e.g.

$$\delta\Gamma^\alpha_{\alpha\rho} \equiv \mathcal{L}_\xi \Gamma^\beta_{\alpha\rho} = \xi^\sigma R^\beta_{\alpha\sigma\rho} + \nabla_\alpha \nabla_\rho \xi^\beta$$

(See also [17] and [18] for a self-contained exposition of the Second Noether Theorem.)

The main contribution to the Noether current comes from the boundary term in (2.6) that when expressed in terms of a new metric (2.9) reads as follow

$$\sqrt{h}\, h^{\alpha\beta}\, (\delta\Gamma^\mu_{\alpha\beta} - \delta^\mu_\beta\, \delta\Gamma^\sigma_{\alpha\sigma})]$$

As a consequence, one obtains the Komar expression

$$U^{\mu\nu}_F(\xi) = |\mathtt{det} h|^{\frac{1}{2}} (\nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu) \tag{2.16}$$

for a superpotential [17, 19, 20, 21, 22] Therefore, an energy-momentum flow as well as a superpotential are already known from the standard Einstein-Hilbert formalism. This extends a notion of universality for the Ricci type Lagrangians also to the energy-momentum complex [3, 20].

3. RELATED DIFFERENTIAL - GEOMETRIC STRUCTURES

The algebraic constraints (2.14) are of special interest by their own. They provide on the space-time some additional differential-geometric structure, namely a metric polynomial structure [23]. A more complete treatment of this subject will be done in a forthcoming publication [15]. For example, a polynomial structure related to the Lagrangians (1.2) is trivial and reduces into $S = I$. Therefore, both metrics g and h coincide and we are left with purely Einstein equations. For the Lagrangians (1.3), a polynomial structure turns out to be well-known a pseudo Riemannian almost-product structure or/and an almost-complex anti-Hermitian ($\equiv$ Norden) structure [3]. Moreover, besides the initial metric g one gets the Einstein metric h. Both metrics are related by algebraic equation $S^2 = \pm I$. This was investigated in [2].

In the (psedo-)Riemannian almost-product case one equivalently deals with an almost-product structure given by the $(1, 1)$ tensor field $S \equiv P$ $(P^2 = I)$ together with a compatible metric h satisfying the condition

$$h(PX, PY) = h(X, Y) \tag{3.1}$$

which is encoded in the simple algebraic relation (2.14). (In our case the metric h should be also Einsteinian.) Here X, Y denote two arbitrary vector fields on M.

There is a wide class of integrable almost-product structures, namely so called *warped product* structures [1, 24], which are an intrinsic property of some well know

exact solutions of Einstein equations: these include e.g. Schwarzschild, Robertson-Walker, Reissner-Nordström, de Sitter, etc. (but not Kerr!). Some other examples are provided by Kaluza-Klein type theories, $3+1$ decompositions and more generally so called *split* structures [25]. The explicit form of the zeta function on product spaces and of the multiplicative anomaly has been derived recently in [26].

In the anti-Hermitian case one deals with $2m$ - dimensional manifold M, an almost complex structure $S \equiv J$ ($J^2 = -I$) and an anti-Hermitian (Norden) metric h [27]: [5]

$$h(JX, JY) = -h(X, Y) \tag{3.2}$$

This implies that the signature of h should be (m, m). In the Kähler-like case ($\nabla J = 0$ for the Levi-Civita connection of h) the almost-complex structure is automatically integrable. We have proved that in fact the metric h has to be a real part of certain holomorphic metric on a complex (space-time) manifold M [2]. This leads to a theory of anti-Kähler manifolds [28].

It should be also remarked that the theory of complex manifolds with holomorphic metric (so called *complex Riemannian* manifolds) has become one of the corner-stone of the twistor theory [29]. This includes a *non-linear graviton* [30], *ambitwistor* formalism [32], theory of H*-spaces* [31] or *Heavens* (i.e. self-dual holomorphic metrics) [32].

Of course, more general Ricci type Lagrangians (2.1) will produce, in general, more complicated Einstein-metric-polynomial structures. For example, the choice $F = s_3^2 \pm 16 s_3$ in $n = 4$ dimensions gives rise to the polynomial equation $S^3 = \mp I$ [15].

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